A Closed-Form Solution for a Gliding Lateral Turn at Constant Height

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This investigation considers the trajectory of an unpowered vehicle executing a lateral turn while maintaining a constant height. Incidence programs are obtained to provide a solution in a closed form. The motion is thought of as consisting of two phases: a constant bank angle phase, matched with appropriate transition phases of increasing and decreasing bank. For the constant bank arc, it is found that there exists a best operating height, with respect to maximum arc length, depending on the particular vehicle configuration and mass. This paper relaxes the conditions of constant angle-of-attack and constant bank angles used in other papers, and thus provides a more flexible approach to the problem of lateral maneuvers.

Nomenclature

A = reference area, ft^2
A_0 = $2mg/\rho NA$, (ft/sec)^2
C_D = drag coefficient
C_{D_0} = profile drag coefficient
C_L = lift coefficient
h = operating height, ft
m = mass, slugs
N = lift curve slope
s = distance along flight path
t = time, sec
V = velocity, fps
W = C_{D_0}/N
x = longitudinal range, ft
y = lateral range, ft
α = incidence, rad
$\hat{\alpha}$ = maximum allowable incidence
β = turn angle, rad
γ = λ^2/V_0^4
λ = $K_1A_0/W^{1/2}$
λ^* = $(2\lambda)^{1/2}$
ρ = atmospheric density, slugs/ft^3
σ = bank angle

Subscripts

0 = initial condition
f = final condition

Superscript

(•) = differentiation with respect to time

Introduction

ONE of the simplest set of equations representing the trajectory of an unpowered lifting vehicle is the case of a lateral turn at constant height over a flat earth using linearized aerodynamic coefficients. This paper gives some closed-form solutions for the aforementioned maneuver.

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The authors are aware of other papers on this topic, but these usually have severe restrictions on bank angle and angle-of-attack. For example, Slye[1] considers flight with a flat earth assumption, but with centrifugal force terms, under the assumption of small inclinations; in order to obtain an analytical solution, the author restricted attention to the equilibrium glide and trajectories with bank angles of nearly 90°. Jackson[2] eliminated some of the disadvantages of Ref. 1 by considering a spherical earth, but in both approaches the assumption of a constant lift/drag ratio was made. Also, Wang[3] considers a lateral maneuver at constant height, but again he restricts his investigation to constant angle-of-attack and bank angles of nearly 90°. However, his paper indicates that for preliminary design work, the flat earth approximation used in this paper is valid.

Let us clearly state what we hope this paper achieves:

1) The angle-of-attack is freed from the constraints previously mentioned.

2) Closed-form solutions are provided for the case of constant bank angle (but the aerodynamic assumption prevents a value of 90° from being realized) and also for a variable bank angle. Flight with variable bank angle can be considered as transition arcs that a) enable the vehicle to attain a constant bank angle from straight and level flight (increasing bank), and b) permit the vehicle to leave the constant bank angle trajectory and return to straight and level flight (decreasing bank).

The closed-form solutions are achieved by suitably choosing the incidence program as a function of velocity. As previously noted, linearized aerodynamic theory with the small-angle approximation is assumed. This assumption demands that the incidence not be permitted to exceed some maximum value consistent with this approximation. In both the constant bank and increasing bank trajectories, this maximum allowable incidence furnished the means of terminating the flight. However, since the decreasing bank arc is flown at constant incidence, the terminating criterion is given by a velocity consideration.

The determination of lateral and longitudinal ranges requires the numerical computation of two integrals. This numerical solution inhibits the derivation of a reasonably easy method of optimizing one of these ranges. But, by considering the total arc length, it is possible for the constant bank case to predict the operating height for a given vehicle configuration at which a maximum arc length can be attained.

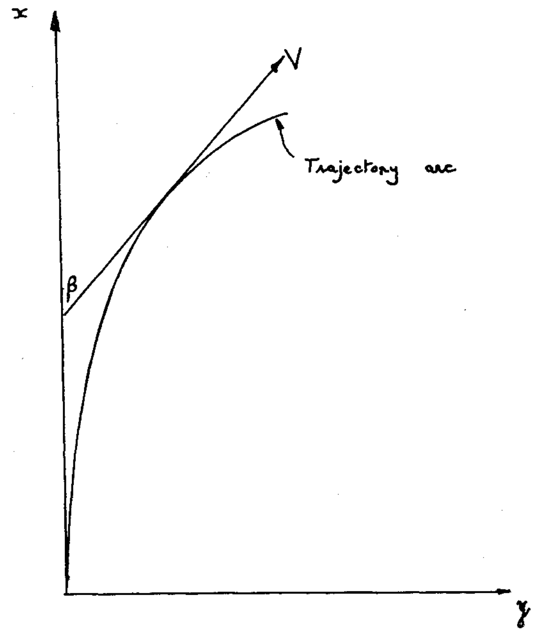

Fig. 1 Trajectory variables.

Analysis

1. Constant Bank Angle Phase

The equations of motion for an unpowered vehicle performing a lateral turn at constant height over a flat earth are,

$$m\dot{V} = \tfrac{1}{2}\rho V^2 A C_D \tag{1}$$

$$mg = \tfrac{1}{2}\rho V^2 A C_L \cos\sigma^* \tag{2}$$

$$mV\dot{\beta} = \tfrac{1}{2}\rho V^2 A C_L \sin\sigma^* \tag{3}$$

where σ^* is the constant bank angle (see Fig. 1).

Considering linearized aerodynamic theory and using the small-angle approximation gives the drag and lift coefficients as,

$$C_D = C_{D_0} + N\alpha^2$$
$$C_L = N\alpha$$

Then Eq. (2) gives the incidence law required to maintain a constant bank flight as

$$\alpha = A_0/V^2 \cos\sigma^* \tag{4}$$

It can be seen that the incidence is inversely proportional to the square of the velocity.

Let $\cos\sigma^* = 1/K_1$, then from (3),

$$d\beta/ds = (\rho AN/2m)(A_0K_1/V^2)(1 - 1/K_1^2)^{1/2} \tag{5}$$

Now from (1),

$$ds/dV = A_0/gV(W + \alpha^2) \tag{6}$$

Substituting (6) into (5) gives

$$d\beta = A_0K_1(1 - 1/K_1^2)^{1/2}dV/V^3(W + \alpha^2)$$

Using the incidence law of (4) and integrating gives the turn angle as a function of velocity:

$$\beta = \frac{1}{2W^{1/2}}\left\{1 - \frac{1}{K_1^2}\right\}\tan^{-1}\left(\frac{V^2}{\lambda}\right) + K_\beta \tag{7}$$

where K_β is a constant of integration.

Similarly, the time integral is obtained as

$$dt = A_0V^2dV/g(WV^4 + A_0^2K_1^2)$$

Hence the time of flight as a function of velocity is

$$t = \frac{A_0}{2gW\lambda^*}\left\{\tan^{-1}\left(\frac{V\lambda^*}{\lambda - V^2}\right) - \frac{1}{2}\log\left(\frac{V^2 + \lambda^*V + \lambda}{V^2 - \lambda^*V + \lambda}\right)\right\} + K_t \tag{8}$$

Now the longitudinal and lateral ranges are given by

$$dx/dt = V\cos\beta \qquad dy/dt = V\sin\beta$$

Hence,

$$x = \frac{A_0}{gW}\int\frac{V^3\cos\beta}{V^4 + \lambda^2}dV \qquad y = \frac{A_0}{gW}\int\frac{V^3\sin\beta}{V^4 + \lambda^2}dV \tag{9}$$

It can be seen that since the parameter K_1 is the inverse of the cosine of the bank angle, then $K_1 \geqslant 1$. Also, the final velocity is restricted by the maximum allowable incidence through the relationship

$$V_f^2 = K_1A_0/\hat{\alpha} \tag{10}$$

The range expressions of Eq. (9) do not lend themselves to an analytic treatment of the problem of maximum range. However, it is possible to develop a procedure that maximizes the arc length and produces some readily usable criteria for sub-optimal performance. It is shown that this approach yields a best operating height, defined as the height corresponding to maximum arc length, for a given vehicle at a particular bank.

Integrating Eq. (6) gives the arc length in terms of velocity as,

$$s = \frac{A_0}{4Wg}\log\left\{\frac{WV_0^4 + K_1^2A_0^2}{WV_f^4 + K_1^2A_0^2}\right\} \tag{11}$$

If V_0 and V_f are given, and remembering the V_f is restricted by (10), the maximum arc length traveled during the velocity interval $V_0 - V_f$ is given by

$$ds/dA_0 = F(V_0) - F(V_f) = 0 \tag{12}$$

where

$$F(V) = \log(V^4 + \lambda^2) + 2\lambda^2/(V^4 + \lambda^2) \tag{13}$$

In Fig. 2, Eq. (12) is solved graphically: the function $F(V)$ is plotted against λ^2 for the velocity range 1000–5000 fps. The intersection of any two curves $F(V_f)$ and $F(V_0)$ gives the required λ^2 for the maximum arc length corresponding to the initial and final velocities of V_0 and V_f, respectively. The value of λ^2 previously obtained gives the optimal value of A_0 for given K_1 and W.

It is not necessary to specify both the initial and final velocities if it can be assumed that the latter is represented by Eq. (10). In this case Eq. (11) can be written independent

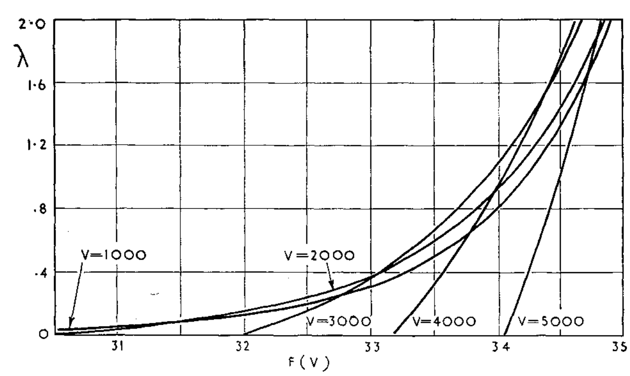

Fig. 2 Constant bank trajectory: plot of λ vs *F*.

of final velocity as,

$$s = (A_0/4Wg)\{\log(\hat{\alpha}^2/\sqrt{W + \hat{\alpha}^2}) + \log\sqrt{(WV_0^4 + K_1^2A_0^2)/K_1^2A_0^2}\}$$

Differentiating with respect to A_0 gives, for maximum arc length,

$$f(\gamma) = \log\sqrt{(W + \hat{\alpha}^2)/\hat{\alpha}^2} \tag{14}$$

where

$$\gamma = \lambda^2/V_0^4$$

$$f = \log\sqrt{(1 + \gamma)/\gamma} - 2/(1 + \gamma)$$

In Fig. 3, the function $f(\gamma)$ is plotted against the parameter W for a range of γ values. The plot of $\log\sqrt{(W + \hat{\alpha}^2)/\hat{\alpha}^2}$ is also presented in Fig. 3; thus the points of intersection give the solution to Eq. (14) and hence the optimal γ for a given W. The optimal A_0 is then easily obtained.

Once the optimal value of A_0 has been determined by one of the aforementioned methods, the best operating height for a given vehicle is obtained from the definition of A_0: $A_0 = 2mg/\rho NA$.

Figures 4a–4c illustrate the method previously developed for the following configuration:

$V_0 = 5{,}000$ $h = 75{,}000$ $\hat{\alpha} = 15°$
$C_{D_0} = 0.25$ $N = 10.87$ $A = 1$
$mg = 750$

2. Increasing Bank Phase

The equations of motion for a variable bank angle are the same as (1–3) except that σ^* is replaced by σ. With this modification, the equations will be denoted $(1)^1$, $(2)^1$, $(3)^1$.

From $(2)^1$, the bank angle is given by

$$\cos\sigma = A_0/V^2\alpha \tag{15}$$

Therefore, for a transition arc of increasing bank, it is necessary to fly an incidence program such that $V^2\alpha$ increases as the velocity decreases because of the effects of drag. It is apparent that the following incidence law satisfies this condition:

$$\alpha = (K_2 - WV^4)^{1/2}/V^2 \tag{16}$$

where K_2 is a constant depending upon initial conditions. It can be noted that many incidence programs would satisfy this primary consideration, but thought must also be given to the ease of integration for the turn angle and time equations. Equation (16) limits the final velocity by the restriction on maximum allowable incidence.

Combining (15) and (16) gives the bank angle program as a function of velocity:

$$\sigma = \cos^{-1}A_0/(K_2 - WV^4)^{1/2} \tag{17}$$

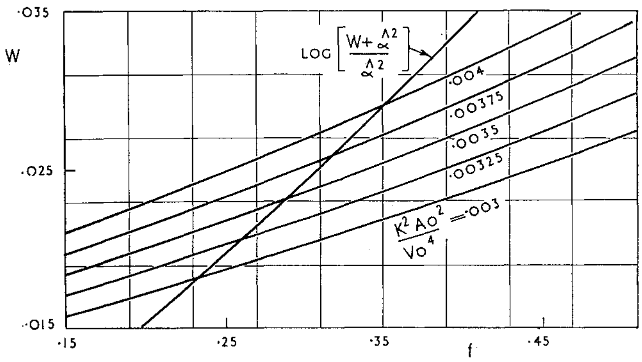

Fig. 3 Constant bank trajectory: plot of W vs f, $\hat{\alpha} = 15°$

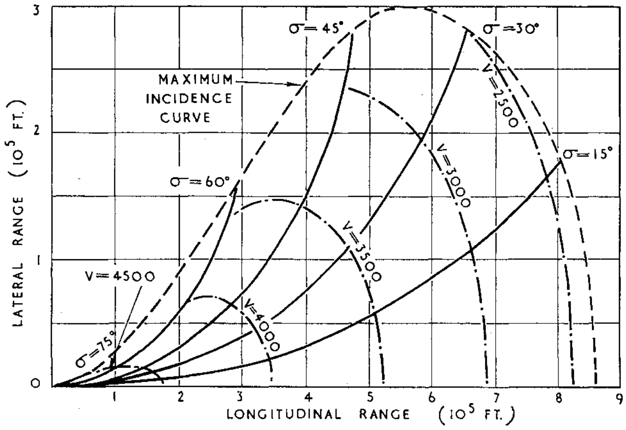

Fig. 4a Constant bank trajectory: constant velocity curves, $\hat{\alpha} = 15°$.

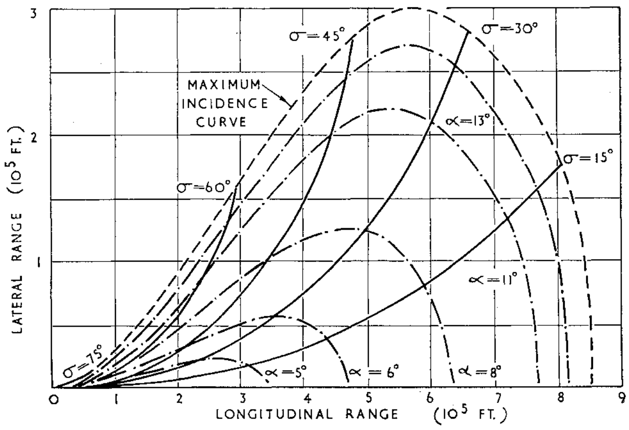

Fig. 4b Constant bank trajectory: constant incidence curves, $\hat{\alpha} = 15°$.

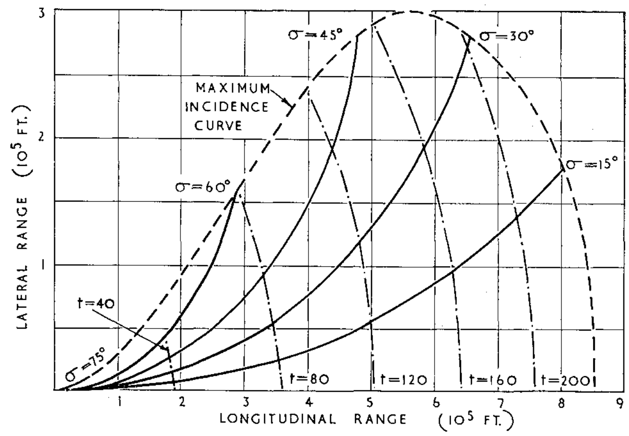

Fig. 4c Constant bank trajectory: constant time curves, $\hat{\alpha} = 15°$.

The other variables are obtained as functions of velocity to give

$$\beta = (W^{1/2}/4K_2)[V^2(\xi^2 - V^4)^{1/2} + \xi^2 \sin^{-1}(V^2/\xi)] + C_\beta \tag{18}$$

where C_β is an integration constant and $\xi^2 = (K_2 - A_0^2)/W$,

$$t = A_0V^3/3K_2g + C_t \tag{19}$$

The value of the parameter K_2 is obtained from the initial conditions as,

$$K_2 = (A_0/\cos\sigma_0)^2 + WV_0^4$$

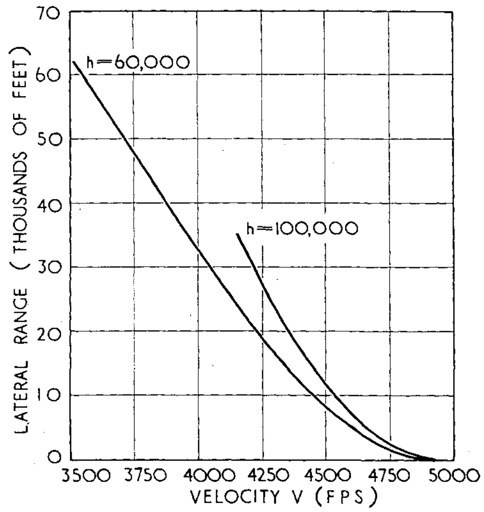

Fig. 5 Increasing bank trajectory: plot of lateral range vs velocity.

Since this trajectory is a transition arc from a lower to a higher bank angle, it is obvious that the rate of change of bank angle is of major importance. From Eqs. (1)[1] and (17) we have

$$d\sigma/dV = 2WV^3 \cos^3\sigma/A_0^2 \sin\sigma \tag{20}$$

$$dV/dt = gK_2/A_0V^2 \tag{21}$$

Hence,

$$d\sigma/dt = (2gK_2W/A_0^3)(V \cos^3\sigma/\sin\sigma)$$

Substituting for K_2 gives

$$\left(\frac{d\sigma}{dt}\right)_0 = \frac{2gW}{A_0^3} V_0 \cot\sigma_0 (A_0 + WV_0^4 \cos^2\sigma_0) \tag{22}$$

Equation (22) indicates that a singularity exists for an initial bank angle of zero. Also, for a given vehicle, smaller initial velocities and larger initial bank angles reduce the rate of change of bank. Alternatively, for the same initial bank and velocity, a reduction in C_{D0} will have the same effect. Figures 5–8 give the trajectory history for heights of 60,000 and 100,000 ft, using an initial bank angle of 1° and the previous configuration.

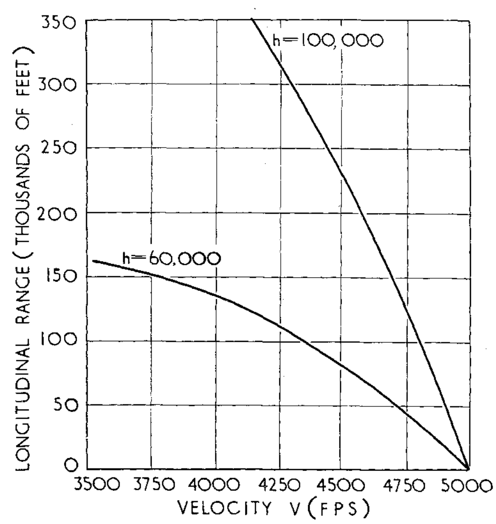

Fig. 6 Increasing bank trajectory: plot of longitudinal range vs velocity.

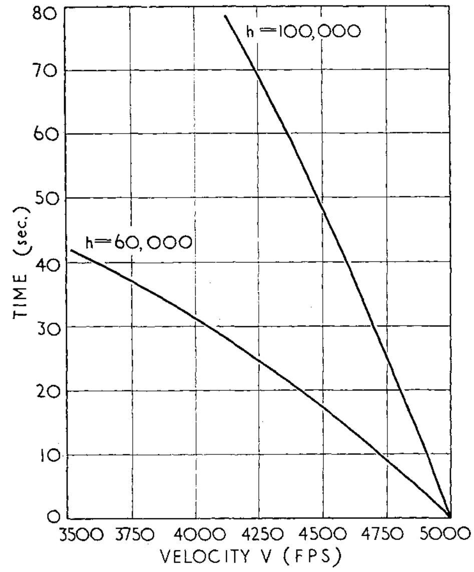

Fig. 7 Increasing bank trajectory: plot of time vs velocity.

3. Decreasing Bank Phase

It has been observed that the maximum incidence limitation is the terminating condition for both the constant and increasing bank phases. Therefore, a constant incidence program is the most obvious choice for a decreasing bank trajectory, since this would allow for smooth transition from either of the previous arcs.

Taking $\alpha = K_3$, where K_3 is a constant depending upon initial conditions, gives the solution for this phase as follows:

$$\beta = (K_3/2\sqrt{W + K^2}) [\log| \tan(\pi/4 + \sigma/2) | - \sin\sigma] \tag{23}$$

$$\sigma = \cos^{-1}A_0/V^2K_3 \tag{24}$$

$$t = A_0/Vg(W + K_3^2) \tag{25}$$

From (24), $K_3 = A_0/V_0^2 \cos\sigma_0$. Therefore, it can be seen that for the same initial bank and velocity, the bank against velocity history is independent of height. The trigonometric condition $\cos\sigma \leqslant 1$ imposes a limit on the final velocity of

$$V_f^2 \geqslant A_0/K_3$$

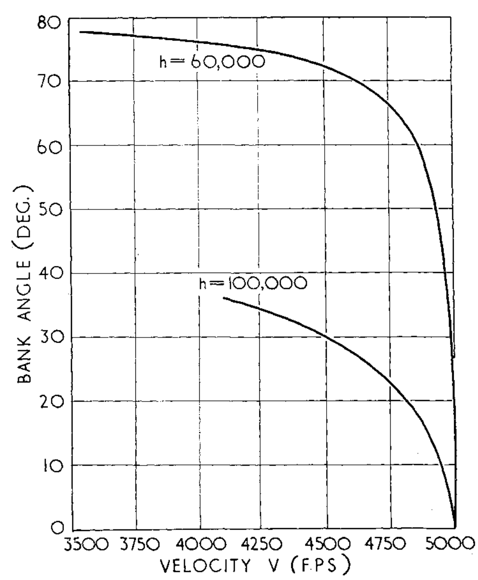

Fig. 8 Increasing bank trajectory: plot of bank angle vs velocity.

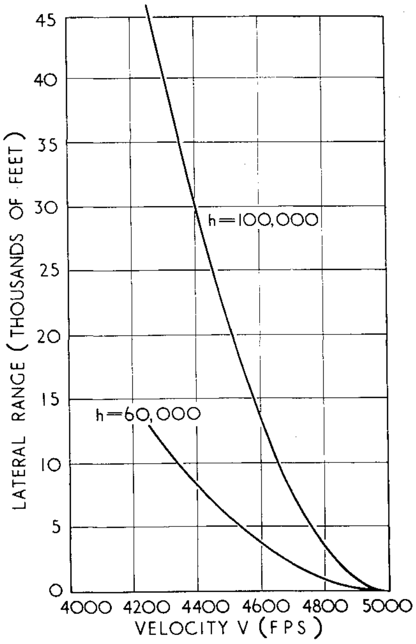

Fig. 9 Decreasing bank trajectory: plot of lateral range vs velocity.

and therefore this expression represents the terminating criterion for the decreasing bank arc.

The rate of change of bank has the same importance here as in the decreasing bank phase. In this case

$$d\sigma/dV = 2A_0/V^3K_3 \sin\sigma \qquad dV/dt = gA_0(W + K_3^2)$$

and therefore,

$$\left(\frac{d\sigma}{dt}\right)_0 = \frac{2A_0^2g(W + K_3^2)}{V_0^3K_3 \sin\sigma_0} \tag{26}$$

Equation (26) again exhibits a singularity at zero bank, but for this case, smaller initial velocities and larger initial bank angles increase the value of $(d\sigma/dt)_0$. A reduction in C_{D_0} for the same initial bank and velocity would have the same effect as in paragraph 2: namely a reduction in $(d\sigma/dt)_0$. Figures 9–12 give the trajectory history for heights of 60,000 and 100,000 ft, using an initial bank angle of 45° and the same configuration as used in paragraph 1.

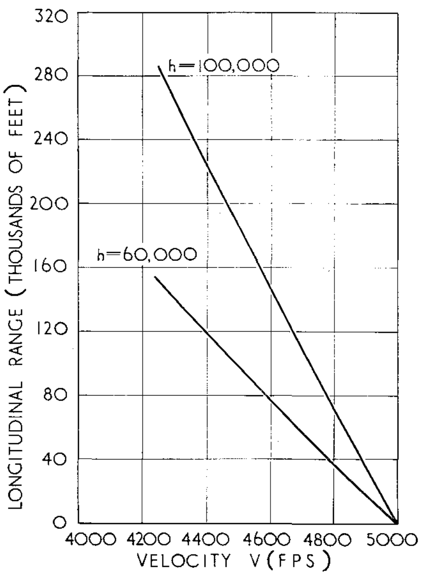

Fig. 10 Decreasing bank trajectory: plot of longitudinal range vs velocity.

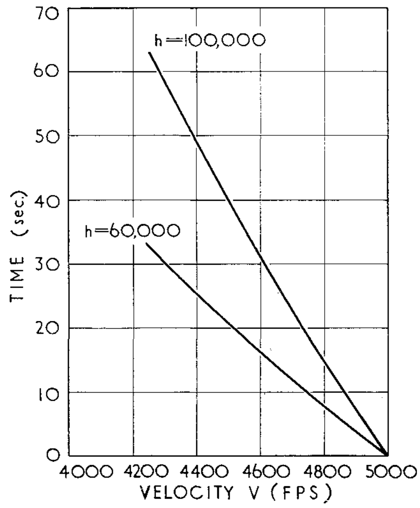

Fig. 11 Decreasing bank trajectory: plot of time vs velocity.

General Discussion of Assumptions and Limitations

The method of this paper uses the flat earth assumption, and therefore its validity is confined to the latter stages of a re-entry maneuver. This restriction is also implicit in the linearized aerodynamic theory assumption, since high hypersonic aerodynamics do not conform with the simple theory used here.

One of the more important limitations of the approach developed in this investigation is the restriction on the final velocity. It can be seen from Eq. (10) that V_f approaches infinity as the bank angle approaches 90°. Thus, one concludes that the method is of value only for bank angles of less than about 70°. The same argument applies to the case of varying bank angle, but with a modification due to the effect of initial velocity.

It can be seen that the expressions representing the rate of change of bank angle with time have singularities at zero bank angle. What does this mean physically and how can it

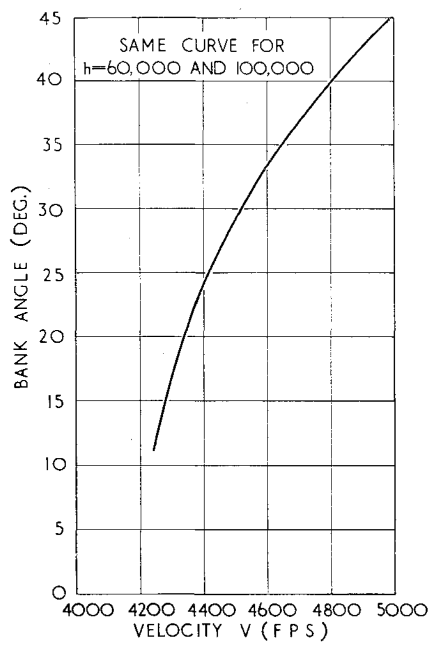

Fig. 12 Decreasing bank trajectory: plot of bank angle vs velocity.

be circumvented? It can be said in answer to the first part of the question that this singular behavior results from the particular choice of bank angle program considered and has no physical significance. The second part of the question is answered by the following discussion.

Let $\cos\sigma = Z(V)$, a function of V; then $-\sin\sigma\, d\sigma/dt = (dZ/dV)\,(dV/dt) = F$, for example. Thus, unless F contains $(1 - Z^2)^{1/2+n}$, where n is a positive integer, there will always exist a singularity at $\sigma = 0$. This is a case of a physical restriction arising from a mathematical assumption.

It would be interesting to compare results from this method with those obtained using other theories. However, as mentioned in the introduction, this theory is essentially one of variable lift/drag ratio and/or variable bank, whereas other approaches are not; thus any comparison between such dissimilar methods would be artificial. We can close this comparison discussion with the comment that the general form of the analysis presented in Pt. 1 would be unaltered if the assumption made by Slye[1] were introduced, namely the addition of a centrifugal term in Eq. (2) but with Eq. (3) the same. The theory would then give an indication of the centrifugal off-loading effects. However, it is not anticipated that this addition would significantly improve the accuracy of the theory over the final phase of the re-entry maneuver, because of the relatively low velocities in this region.

Conclusions

The problem of unpowered flight at a constant height has been solved in closed form for the three fundamentally different cases of increasing bank, constant bank, decreasing bank. These trajectories should find application in preliminary design studies for re-entry vehicles, although only a simple aerodynamic theory has been used. In addition, a method has been presented for choosing a best operating height for a particular vehicle on a constant bank trajectory.

References

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